

ALGEBRAIC INDEPENDENCE RESULTS FOR VALUES OF THETA-CONSTANTS, II

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Dedicated to Professor Iekata Shiokawa on the occasion of his 75th birthday

Abstract

Let $\theta_3(\tau) = 1 + 2 \sum_{\nu=1}^{\infty} q^{\nu^2}$ with $q = e^{i\pi\tau}$ denote the Thetanullwert of the Jacobi theta function

$$\theta(z|\tau) = \sum_{\nu=-\infty}^{\infty} e^{\pi i \nu^2 \tau + 2\pi i \nu z}.$$

Moreover, let $\theta_2(\tau) = 2 \sum_{\nu=0}^{\infty} q^{(\nu+1/2)^2}$ and $\theta_4(\tau) = 1 + 2 \sum_{\nu=1}^{\infty} (-1)^{\nu} q^{\nu^2}$. For algebraic numbers q with $0 < |q| < 1$ and for any $j \in \{2, 3, 4\}$ we prove the algebraic independence over \mathbb{Q} of the numbers $\theta_j(n\tau)$ and $\theta_j(\tau)$ for all odd integers $n \geq 3$. Assuming the same conditions on q and τ as above, we obtain sufficient conditions by use of a criterion involving resultants in order to decide on the algebraic independence over \mathbb{Q} of $\theta_j(2m\tau)$ and $\theta_j(\tau)$ ($j = 2, 3, 4$) and of $\theta_3(4m\tau)$ and $\theta_3(\tau)$ with odd positive integers m . In particular, we prove the algebraic independence of $\theta_3(n\tau)$ and $\theta_3(\tau)$ for even integers n with $2 \leq n \leq 22$. The paper continues the work of the first-mentioned author, who already proved the algebraic independence of $\theta_3(2^m\tau)$ and $\theta_3(\tau)$ for $m = 1, 2, \dots$

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1 Introduction and statement of results

Let τ be a complex variable in the complex upper half-plane $\Im(\tau) > 0$. The series

$$\theta_2(\tau) = 2 \sum_{\nu=0}^{\infty} q^{(\nu+1/2)^2}, \quad \theta_3(\tau) = 1 + 2 \sum_{\nu=1}^{\infty} q^{\nu^2}, \quad \theta_4(\tau) = 1 + 2 \sum_{\nu=1}^{\infty} (-1)^{\nu} q^{\nu^2}$$

are known as theta-constants or Thetanullwerte, where $q = e^{\pi i \tau}$. In particular, $\theta_3(\tau)$ is the Thetanullwert of the Jacobi theta function $\theta(z|\tau) = \sum_{\nu=-\infty}^{\infty} e^{\pi i \nu^2 \tau + 2\pi i \nu z}$. For an extensive discussion of theta-functions and theta-constants we refer the reader to [4], [5], and [6]. Recently, the first-named author has proven the following result.

Theorem A. [3, Theorem 1.1] *Let q be an algebraic number with $q = e^{\pi i \tau}$ and $\Im(\tau) > 0$. Let $m \geq 1$ be an integer. Then, the two numbers $\theta_3(2^m \tau)$ and $\theta_3(\tau)$ are algebraically independent over \mathbb{Q} as well as the two numbers $\theta_3(n\tau)$ and $\theta_3(\tau)$ for $n = 3, 5, 6, 7, 9, 10, 11, 12$.*

The first basic tool in proving such algebraic independence results are integer polynomials in two variables X, Y , which vanish at certain points $X = X_0$ and $Y = Y_0$ given by values of rational functions of theta-constants. For instance, for $n = 2^m$ ($m \geq 3$) we consider the polynomial

$$P_n(X, Y) = (nX - (1 + Y)^2)^{2^{m-2}} + Y U_n(X, (1 + Y)^2, Y)$$

where $U_n(t_1, t_2, t_3) \in \mathbb{Q}[t_1, t_2, t_3]$ is a polynomial satisfying

$$U_n\left(\frac{1}{n}, 1, 0\right) = -2^{2^{m-1}-1}.$$

Moreover, we have

$$P_n\left(\frac{\theta_3^2(n\tau)}{\theta_3^2(\tau)}, \frac{\theta_4(\tau)}{\theta_3(\tau)}\right) = 0.$$

The second tool is an algebraic independence criterion (see Lemma 2.5 below), from which the algebraic independence of $\theta_3(n\tau)$ and $\theta_3(\tau)$ over \mathbb{Q} can be obtained by proving that the resultant

$$\text{Res}_X \left(P_n(X, Y), \frac{\partial}{\partial Y} P_n(X, Y) \right) \in \mathbb{Z}[Y]$$

does not vanish identically (see [3, Theorem 4.1]). This can be seen as follows. We have the identities

$$\begin{aligned} P_n(X, 0) &= (2^m X - 1)^{2^{m-2}}, \\ \frac{\partial P_n}{\partial Y}(X, 0) &= -2^{m-1} (2^m X - 1)^{2^{m-2}-1} + U_n(X, 1, 0), \end{aligned}$$

from which on the one hand we deduce that $P_n(X, 0)$ has a 2^{m-2} -fold root at $X_1 = 1/2^m$. On the other hand one has

$$\frac{\partial P_n}{\partial Y}(X_1, 0) = U_n\left(\frac{1}{n}, 1, 0\right) = -2^{2^{m-1}-1} \neq 0.$$

Hence, for $Y = 0$ the polynomials $P_n(X, Y)$ and $\partial P_n(X, Y)/\partial Y$ have no common root. Therefore, the above resultant with respect to X does not vanish identically, which gives the desired result. We state

the polynomials $P_{2^m}(X, Y)$ for $m = 1, 2, 3, 4$ explicitly. The algorithm to compute these polynomials recursively is given by Lemma 3.1 in [3].

$$\begin{aligned} P_2 &= 2X - Y^2 - 1, \\ P_4 &= 4X - (1 + Y)^2, \\ P_8 &= 64X^2 - 16(1 + Y)^2X + (1 - Y)^4, \\ P_{16} &= 65536X^4 - 16384(1 + Y)^2X^3 + 512(3Y^4 + 4Y^3 + 18Y^2 + 4Y + 3)X^2 \\ &\quad - 64(1 + Y)^2(Y^4 + 28Y^3 + 6Y^2 + 28Y + 1)X + (1 - Y)^8, \end{aligned}$$

Let $n \geq 3$ denote an odd positive integer. Set

$$h_j(\tau) := n^2 \frac{\theta_j^4(n\tau)}{\theta_j^4(\tau)} \quad (j = 2, 3, 4), \quad \lambda = \lambda(\tau) := \frac{\theta_2^4(\tau)}{\theta_3^4(\tau)}, \quad \psi(n) := n \prod_{p|n} \left(1 + \frac{1}{p}\right),$$

where p runs through all primes dividing n . Yu.V. Nesterenko [8] proved the existence of integer polynomials $P_n(X, Y) \in \mathbb{Z}[X, Y]$ such that $P_n(h_j(\tau), R_j(\lambda(\tau))) = 0$ holds for $j = 2, 3, 4$, odd integers $n \geq 3$, and a suitable rational function R_2, R_3 , or R_4 , respectively.

Theorem B. [8, Theorem 1, Corollary 3] *For any odd integer $n \geq 3$ there exists a polynomial $P_n(X, Y) \in \mathbb{Z}[X, Y]$, $\deg_X P_n = \psi(n)$, such that*

$$P_n\left(h_2(\tau), 16 \frac{\lambda(\tau) - 1}{\lambda(\tau)}\right) = 0, \quad (1.1)$$

$$P_n(h_3(\tau), 16\lambda(\tau)) = 0, \quad (1.2)$$

$$P_n\left(h_4(\tau), 16 \frac{\lambda(\tau)}{\lambda(\tau) - 1}\right) = 0. \quad (1.3)$$

The polynomials P_3, P_5, P_7, P_9 , and P_{11} are listed in the appendix of [3]. P_3 and P_5 are already given in [8], P_7, P_9 , and P_{11} are the results of computer-assisted computations of the first-named author.

In this paper we focus on the problem to decide on the algebraic independence of $\theta_j(n\tau)$ and $\theta_j(\tau)$ ($j = 2, 3, 4$) over \mathbb{Q} for algebraic numbers q , where $n \geq 3$ is an odd integer or $n = 2m, 4m$ with odd positive integers m . The above Theorem B will be used in Section 2.

In the following theorems, the number $q = e^{\pi i \tau}$ is an algebraic number with $\Im(\tau) > 0$.

Theorem 1.1. *Let $n \geq 3$ be an odd integer. Then, the numbers in each of the sets*

$$\{\theta_2(n\tau), \theta_2(\tau)\}, \quad \{\theta_3(n\tau), \theta_3(\tau)\}, \quad \{\theta_4(n\tau), \theta_4(\tau)\}$$

are algebraically independent over \mathbb{Q} .

In order to prove this theorem we first shall show that for an algebraic number q with $0 < |q| < 1$ the numbers $h_2(\tau)$, $h_3(\tau)$, and $h_4(\tau)$ are transcendental (Lemma 2.3). This interim result already shows that the two numbers $\theta_j(n\tau)$ and $\theta_j(\tau)$ ($j = 2, 3, 4$) are homogeneously algebraically independent over \mathbb{Q} .

On the other hand, it has not been shown that Theorem 1.1 holds for arbitrary even integers n . However we can prove it for small even integers n by checking the non-vanishing of a Jacobian determinant (Lemma 2.5), which is hard to decide when the involved polynomials are not given explicitly.

Theorem 1.2. For $n = 2, 4, 6$, the numbers $\theta_2(n\tau)$ and $\theta_2(\tau)$ are algebraically independent over \mathbb{Q} .

Theorem 1.3. For $n = 2, 4, 6, 8, 10$, the numbers $\theta_4(n\tau)$ and $\theta_4(\tau)$ are algebraically independent over \mathbb{Q} .

Theorem 1.4. Let $2 \leq n \leq 22$ be an even integer. Then, the numbers $\theta_3(n\tau)$ and $\theta_3(\tau)$ are algebraically independent over \mathbb{Q} .

2 Auxiliary results

In this section, we prepare some lemmas to prove theorems.

Lemma 2.1. [2, Lemma 4] Let q be an algebraic number with $q = e^{\pi i \tau}$ and $\Im(\tau) > 0$. Then, any two numbers in the set in each of the sets

$$\{\theta_2(\tau), \theta_3(\tau), \theta_4(\tau)\}$$

are algebraically independent over \mathbb{Q} .

This result can be derived from Yu.V. Nesterenko's theorem [7] on the algebraic independence of the values $P(q), Q(q), R(q)$ of the Ramanujan functions P, Q, R at a nonvanishing algebraic point q . It should be noticed that the three numbers $\theta_2(\tau), \theta_3(\tau)$, and $\theta_4(\tau)$ are algebraically dependent over \mathbb{Q} , since the identity

$$\theta_3^4(\tau) = \theta_2^4(\tau) + \theta_4^4(\tau) \quad (2.1)$$

holds for any $\tau \in \mathbb{C}$ with $\Im(\tau) > 0$.

In what follows, we distinguish two cases based on the parity of n .

2.1 The case where n is odd

The following subsequent Lemmas 2.2, 2.3, and 2.4 are needed to prove Theorem 1.1. Let $n \geq 3$ be a fixed odd integer and $\tau \in \mathbb{C}$ with $\Im(\tau) > 0$. From Theorem B we know that there exists a nonzero polynomial $P_n(X, Y) \in \mathbb{Z}[X, Y]$ with $\deg_X P_n = \psi(n)$ such that $P_n(X_0, Y_0)$ vanishes for

$$X_0 := h_3(\tau) = n^2 \frac{\theta_3^4(n\tau)}{\theta_3^4(\tau)}, \quad Y_0 := 16\lambda(\tau) = 16 \frac{\theta_2^4(\tau)}{\theta_3^4(\tau)}.$$

Let $N := \deg_Y P_n(X, Y)$. The polynomials $Q_j(X) \in \mathbb{Z}[X]$ ($j = 0, 1, \dots, N$) are given by

$$P_n(X, Y) = \sum_{j=0}^N Q_j(X) Y^j. \quad (2.2)$$

Lemma 2.2. For any complex number α , there exists a subscript $j = j(\alpha)$ such that $Q_j(\alpha) \neq 0$.

Proof of Lemma 2.2. Suppose on the contrary that there exists an $\alpha \in \mathbb{C}$ such that $Q_j(\alpha) = 0$ for all $j = 0, 1, \dots, N$. It follows from (2.2) that there exists a polynomial $R_n(X, Y) \in \mathbb{C}[X, Y]$ satisfying

$$P_n(X, Y) = (X - \alpha) R_n(X, Y). \quad (2.3)$$

In accordance with formula (5) in [8] we define for any $\tau \in \mathbb{C}$ with $\Im(\tau) > 0$ the numbers

$$x_\nu(\tau) := u^2 \frac{\theta_3^4\left(\frac{u\tau+2v}{w}\right)}{\theta_3^4(\tau)} \quad (\nu = 1, 2, \dots, \psi(n)),$$

where the nonnegative integers u, v, w are given by [8, Lemma 1]. These integers depend on n and ν and satisfy the three conditions

$$(u, v, w) = 1, \quad uw = n, \quad 0 \leq v < w. \quad (2.4)$$

Substituting $Y = 16\lambda(\tau)$ into (2.3), we have by [8, Corollary 1]

$$\prod_{\nu=1}^{\psi(n)} (X - x_\nu(\tau)) = P_n(X, 16\lambda(\tau)) = (X - \alpha)R_n(X, 16\lambda(\tau)). \quad (2.5)$$

Next, by substituting $X = \alpha$ into (2.5), we obtain

$$\prod_{\nu=1}^{\psi(n)} (\alpha - x_\nu(\tau)) = 0 \quad (2.6)$$

for any $\tau \in \mathbb{C}$ with $\Im(\tau) > 0$. Let $a_k := nki$ ($k = 1, 2, \dots$) be a sequence of complex numbers on the imaginary axis. Then we get by (2.6)

$$\prod_{\nu=1}^{\psi(n)} (\alpha - x_\nu(a_k)) = 0 \quad (k = 1, 2, \dots).$$

Hence, by the pigeonhole principle, there is a subscript ν_0 with $1 \leq \nu_0 \leq \psi(n)$ such that

$$u^2 \frac{\theta_3^4\left(\frac{ub_k+2v}{w}\right)}{\theta_3^4(b_k)} = x_{\nu_0}(b_k) = \alpha \quad (2.7)$$

holds for some subsequence $\{b_k\}_{k \geq 1}$ of $\{a_k\}_{k \geq 1}$. The integers u, v, w in (2.7) depend on n, ν_0 and satisfy the conditions in (2.4). Since $\sqrt[4]{\alpha/u^2}$ takes four complex values, we see in the same way by applying the pigeonhole principle that there exists a complex number β with $\beta^4 = \alpha/u^2$ such that

$$\frac{\theta_3\left(\frac{uc_k+2v}{w}\right)}{\theta_3(c_k)} = \beta \quad (2.8)$$

holds for some subsequence $\{c_k\}_{k \geq 1}$ of $\{b_k\}_{k \geq 1}$. Let $c_k := nt_k i$, where t_k ($k \geq 1$) are positive integers with $t_1 < t_2 < \dots$. Then, by substituting $\tau = c_k$ into $q = e^{\pi i \tau} = e^{-\pi t_k n}$, we obtain

$$\theta_3(c_k) = 1 + 2 \sum_{m=1}^{\infty} (e^{-\pi t_k})^{nm^2}. \quad (2.9)$$

For $\tau = \frac{uc_k+2v}{w}$ it follows with $\xi_w := e^{\frac{2\pi i}{w}}$ that

$$q = e^{\pi i \frac{unt_k i + 2v}{w}} = \left(e^{\frac{2\pi i}{w}}\right)^v \cdot e^{-\pi u^2 t_k} = \xi_w^v \cdot e^{-\pi u^2 t_k},$$

which yields

$$\theta_3\left(\frac{uc_k+2v}{w}\right) = 1 + 2 \sum_{m=1}^{\infty} \xi_w^{vm^2} (e^{-\pi t_k})^{u^2 m^2}. \quad (2.10)$$

Next, two complex functions $f(z)$ and $g(z)$ are defined by their power series, namely

$$\begin{aligned} f(z) &:= 1 + 2 \sum_{m=1}^{\infty} z^{nm^2}, \\ g(z) &:= 1 + 2 \sum_{m=1}^{\infty} \xi_w^{vm^2} z^{u^2m^2}. \end{aligned}$$

Setting $\eta_k := e^{-\pi t_k}$ ($k = 1, 2, \dots$), it follows from (2.8) to (2.10) that

$$\beta f(\eta_k) = g(\eta_k) \quad (k = 1, 2, \dots).$$

The sequence $\{\eta_k\}_{k \geq 1}$ tends to zero, such that we may apply the identity theorem for power series. We obtain

$$\beta f(z) = g(z) \quad (|z| < 1).$$

Comparing the first and second nonvanishing coefficients of the series, it follows that $\beta = 1$, and, by applying $uw = n$ in (2.4), $u^2 = n$, $u = w = \sqrt{n} > 1$, $\xi_w^v = 1$. Since $\xi_w \neq 1$, we conclude from $\xi_w^v = 1$ and $0 \leq v < w$ in (2.4) that $v = 0$. Finally, we deduce that $(u, v, w) = (u, 0, u) = u > 1$, which contradicts the arithmetic condition $(u, v, w) = 1$ in (2.4). This completes the proof of Lemma 2.2. \square

Lemma 2.3. *Let q be an algebraic number with $q = e^{\pi i \tau}$ and $\Im(\tau) > 0$. Then the three numbers $h_2(\tau)$, $h_3(\tau)$, and $h_4(\tau)$ are transcendental.*

Proof of Lemma 2.3. We suppose on the contrary that $h_3(\tau)$ is an algebraic number. Then it follows from (2.2) and from Lemma 2.2 that

$$F(Y) := P_n(h_3(\tau), Y) = \sum_{j=0}^N Q_j(h_3(\tau)) Y^j$$

is a nonzero polynomial with algebraic coefficients. Hence, the identity (1.2) yields $F(16\lambda(\tau)) = 0$, thus showing that $\lambda(\tau) = \frac{\theta_2^4(\tau)}{\theta_3^4(\tau)}$ is an algebraic number. But, by Lemma 2.1, the numbers $\theta_2(\tau)$ and $\theta_3(\tau)$ are algebraically independent over \mathbb{Q} , a contradiction. Thus, $h_3(\tau)$ is transcendental. Similarly, the transcendence of $h_2(\tau)$ and $h_4(\tau)$ follows from the identities (1.1) and (1.3). \square

Lemma 2.4. *Let $\alpha_1, \alpha_2 \in \mathbb{C}$ be algebraically independent over \mathbb{Q} and let $\beta_1, \beta_2 \in \mathbb{C}$ ($\beta_2 \neq 0$) such that β_1/β_2 is transcendental. Suppose that there exist nonzero polynomials $P(X, Y), Q(X, Y) \in \mathbb{Q}[X, Y]$ such that*

$$P(\alpha_1/\alpha_2, \beta_1/\beta_2) = 0. \quad (2.11)$$

and

$$Q(\beta_1, \beta_2) = \alpha_2. \quad (2.12)$$

Then β_1 and β_2 are algebraically independent over \mathbb{Q} .

Proof of Lemma 2.4. Define the fields $F := \mathbb{Q}(\beta_1, \beta_2)$ and $E := F(\alpha_1, \alpha_2)$. We first prove that the field extension E/F is algebraic. Since $\alpha_2 \in F$ by (2.12), we only have to show that α_1 is algebraic over F . Let

$$P(X, Y) := \sum_{j=0}^{\ell} R_j(Y) X^j, \quad R_j(Y) \in \mathbb{Q}[Y], \quad R_{\ell}(Y) \neq 0,$$

and define

$$f(X) := \sum_{j=0}^{\ell} R_j(\beta_1/\beta_2)(X/\alpha_2)^j \in F[X],$$

where $f(X)$ is a nonzero polynomial, since $R_{\ell}(\beta_1/\beta_2) \neq 0$ follows from the transcendence of β_1/β_2 . By (2.11), we have $f(\alpha_1) = P(\alpha_1/\alpha_2, \beta_1/\beta_2) = 0$, which implies that α_1 is algebraic over F .

Thus we get

$$\text{trans. deg } F/\mathbb{Q} = \text{trans. deg } E/F + \text{trans. deg } F/\mathbb{Q} = \text{trans. deg } E/\mathbb{Q} \geq 2,$$

where we used the algebraic independence hypothesis on α_1 and α_2 . On the other hand, $\text{trans. deg } F/\mathbb{Q} \leq 2$ is trivial. Therefore we obtain

$$\text{trans. deg } F/\mathbb{Q} = 2,$$

which gives the desired result. \square

2.2 The case where n is even

We need the expressions of $\theta_j(2^\ell \tau)$ ($j = 2, 3, 4, \ell = 1, 2, 3$) in terms of $\theta_j := \theta_j(\tau)$ ($j = 2, 3, 4$), which will be used to prove Theorems 1.2, 1.3, and 1.4. Recall that the following identities hold for any $\tau \in \mathbb{C}$ with $\Im(\tau) > 0$:

$$2\theta_2^2(2\tau) = \theta_3^2 - \theta_4^2, \quad (2.13)$$

$$2\theta_3^2(2\tau) = \theta_3^2 + \theta_4^2, \quad (2.14)$$

$$\theta_4^2(2\tau) = \theta_3\theta_4, \quad (2.15)$$

and

$$2\theta_2(4\tau) = \theta_3 - \theta_4, \quad (2.16)$$

$$2\theta_3(4\tau) = \theta_3 + \theta_4, \quad (2.17)$$

$$2\theta_4^4(4\tau) = (\theta_3^2 + \theta_4^2)\theta_3\theta_4. \quad (2.18)$$

The most important tool to transfer the algebraic independence of a set of m numbers to another set of m numbers, which all satisfy a system of algebraic identities, is given by the following lemma. We call it an *algebraic independence criterion* (AIC).

Lemma 2.5. [1, Lemma 3.1] *Let $x_1, \dots, x_m \in \mathbb{C}$ be algebraically independent over \mathbb{Q} and let $y_1, \dots, y_m \in \mathbb{C}$ satisfy the system of equations*

$$f_j(x_1, \dots, x_m, y_1, \dots, y_m) = 0 \quad (1 \leq j \leq m),$$

where $f_j(t_1, \dots, t_m, u_1, \dots, u_m) \in \mathbb{Q}[t_1, \dots, t_m, u_1, \dots, u_m]$ ($1 \leq j \leq m$). Assume that

$$\det \left(\frac{\partial f_j}{\partial t_i}(x_1, \dots, x_m, y_1, \dots, y_m) \right) \neq 0.$$

Then the numbers y_1, \dots, y_m are algebraically independent over \mathbb{Q} .

We shall apply the AIC to the sets $\{x_1, x_2\}$ with $x_1, x_2 \in \mathbb{Z}[\theta_2, \theta_3, \theta_4]$.

2.2.1 The case $n = 2m$ with odd integer m

In this subsection, we put $n = 2m$ with an odd integer $m > 1$. In Lemmas 2.6, 2.7, and 2.8 below, we give sufficient conditions for the numbers in each of the set $\{\theta_j(n\tau), \theta_j(\tau)\}$ ($j = 2, 3, 4$) to be algebraically independent over \mathbb{Q} . Replacing τ by 2τ in (1.1), (1.2), and (1.3), we have

$$P_m(X_0, Y_0) = 0$$

for

$$X_0 = h_2(2\tau) = m^2 \frac{\theta_2^4(n\tau)}{\theta_2^4(2\tau)} \quad \text{and} \quad Y_0 = 16 \frac{\lambda(2\tau) - 1}{\lambda(2\tau)} = 16 \frac{\theta_2^4(2\tau) - \theta_3^4(2\tau)}{\theta_2^4(2\tau)}, \quad (2.19)$$

$$X_0 = h_3(2\tau) = m^2 \frac{\theta_3^4(n\tau)}{\theta_3^4(2\tau)} \quad \text{and} \quad Y_0 = 16\lambda(2\tau) = 16 \frac{\theta_2^4(2\tau)}{\theta_3^4(2\tau)}, \quad (2.20)$$

$$X_0 = h_4(2\tau) = m^2 \frac{\theta_4^4(n\tau)}{\theta_4^4(2\tau)} \quad \text{and} \quad Y_0 = 16 \frac{\lambda(2\tau)}{\lambda(2\tau) - 1} = 16 \frac{\theta_2^4(2\tau)}{\theta_2^4(2\tau) - \theta_3^4(2\tau)}, \quad (2.21)$$

respectively.

Let q be an algebraic number with $q = e^{\pi i \tau}$ and $\Im(\tau) > 0$. The total degree of $P_m(X, Y)$ is denoted by M .

Lemma 2.6. *If the polynomial*

$$\text{Res}_X \left(P_m(X, Y), X \frac{\partial}{\partial X} P_m(X, Y) + 2(Y - 16) \frac{\partial}{\partial Y} P_m(X, Y) \right)$$

does not vanish identically, then the numbers $\theta_2(n\tau)$ and $\theta_2(\tau)$ are algebraically independent over \mathbb{Q} .

Proof of Lemma 2.6. Let

$$\begin{aligned} x_1 &:= (\theta_3^4 - \theta_4^4)^2, & x_2 &:= (\theta_3^2 + \theta_4^2)^2, \\ y_1 &:= 4m^2 \theta_2^4(n\tau), & y_2 &:= \theta_2^4. \end{aligned}$$

Then the numbers x_1 and x_2 are algebraically independent over \mathbb{Q} . Indeed, the numbers θ_3 and θ_4 are the roots of polynomial

$$T^8 - \frac{1}{2} \left(\frac{x_1}{x_2} + x_2 \right) T^4 + \frac{1}{16} \left(\frac{x_1}{x_2} - x_2 \right)^2,$$

so that the field $E := \mathbb{Q}(\theta_3, \theta_4)$ is an algebraic extension of $F := \mathbb{Q}(x_1, x_2)$, and hence by Lemma 2.1

$$\text{trans. deg } F/\mathbb{Q} = \text{trans. deg } E/F + \text{trans. deg } F/\mathbb{Q} = \text{trans. deg } E/\mathbb{Q} = 2.$$

By the identities (2.13) and (2.14), the numbers X_0 and Y_0 in (2.19) are expressed as

$$X_0 = \frac{x_2 y_1}{x_1} \quad \text{and} \quad Y_0 = 16 \frac{x_1 - x_2^2}{x_1}.$$

Define

$$\begin{aligned}
 g_1(t_1, t_2, u_1, u_2) &:= \frac{t_2 u_1}{t_1}, \\
 g_2(t_1, t_2, u_1, u_2) &:= 16 \frac{t_1 - t_2^2}{t_1}, \\
 f_1(t_1, t_2, u_1, u_2) &:= t_1^M P_m(g_1, g_2) \\
 f_2(t_1, t_2, u_1, u_2) &:= u_2^2 - t_1.
 \end{aligned} \tag{2.22}$$

Since $g_1(x_1, x_2, y_1, y_2) = X_0$ and $g_2(x_1, x_2, y_1, y_2) = Y_0$,

$$f_1(x_1, x_2, y_1, y_2) = x_1^M P_m(X_0, Y_0) = 0$$

and by the identity (2.1)

$$f_2(x_1, x_2, y_1, y_2) = y_2^2 - x_1 = 0.$$

Using the algebraic independence criterion we have to show the nonvanishing of

$$\Delta := \det \begin{pmatrix} \frac{\partial f_1}{\partial t_1} & \frac{\partial f_1}{\partial t_2} \\ \frac{\partial f_2}{\partial t_1} & \frac{\partial f_2}{\partial t_2} \end{pmatrix} = \frac{\partial f_1}{\partial t_2}$$

at (x_1, x_2, y_1, y_2) ; namely by (2.22)

$$\frac{\partial f_1}{\partial t_2}(x_1, x_2, y_1, y_2) = x_1^M \frac{\partial P_m}{\partial t_2}(X_0, Y_0) \neq 0.$$

Applying the chain rule, we obtain

$$\begin{aligned}
 \frac{\partial P_m}{\partial t_2}(X_0, Y_0) &= \frac{\partial P_m}{\partial X}(X_0, Y_0) \cdot \frac{\partial g_1}{\partial t_2}(x_1, x_2, y_1, y_2) + \frac{\partial P_m}{\partial Y}(X_0, Y_0) \cdot \frac{\partial g_2}{\partial t_2}(x_1, x_2, y_1, y_2) \\
 &= \frac{1}{x_2} \left(X_0 \frac{\partial P_m}{\partial X}(X_0, Y_0) + 2(Y_0 - 16) \frac{\partial P_m}{\partial Y}(X_0, Y_0) \right).
 \end{aligned}$$

Therefore, in order to prove the lemma by the algebraic independence criterion, it suffices to show that

$$X_0 \frac{\partial P_m}{\partial X}(X_0, Y_0) + 2(Y_0 - 16) \frac{\partial P_m}{\partial Y}(X_0, Y_0) \neq 0. \tag{2.23}$$

By the hypothesis of Lemma 2.6 the polynomial

$$R(Y) := \text{Res}_X \left(P_m(X, Y), X \frac{\partial}{\partial X} P_m(X, Y) + 2(Y - 16) \frac{\partial}{\partial Y} P_m(X, Y) \right) \in \mathbb{Z}[Y]$$

does not vanish identically. For fixed $Y = Y_0 := 16(x_1 - x_2^2)/x_1$ we have $R(Y_0) \in \mathbb{Q}(x_1, x_2)$, so that the algebraic independence of x_1, x_2 proves $R(Y_0) \neq 0$. In particular, $P_m(X, Y_0)$ and

$$X \frac{\partial}{\partial X} P_m(X, Y_0) + 2(Y_0 - 16) \frac{\partial}{\partial Y} P_m(X, Y_0)$$

(which both are polynomials in X) have no common root. Since $P_m(X, Y_0)$ vanishes for $X = X_0 := x_2 y_1 / x_1$, we obtain (2.23). The proof of Lemma 2.6 is completed. \square

Lemma 2.7. *If the polynomial*

$$\text{Res}_X \left(P_m(X^2, Y^2), X^2 \frac{\partial}{\partial X} P_m(X^2, Y^2) + (Y^2 + 4Y) \frac{\partial}{\partial Y} P_m(X^2, Y^2) \right)$$

does not vanish identically, then the numbers $\theta_3(n\tau)$ and $\theta_3(\tau)$ are algebraically independent over \mathbb{Q} .

Proof of Lemma 2.7. Let

$$\begin{aligned} x_1 &:= 2\theta_3^2, & x_2 &:= \theta_3^2 + \theta_4^2, \\ y_1 &:= 2m\theta_3^2(n\tau), & y_2 &:= \theta_3^2. \end{aligned}$$

Then the numbers x_1, x_2 are algebraically independent over \mathbb{Q} and we see by (2.13) and (2.14) that the numbers X_0 and Y_0 in (2.20) are given by

$$X_0 = \frac{y_1^2}{x_2^2} \quad \text{and} \quad Y_0 = (\sqrt{Y_0})^2 := \left(\frac{4(x_1 - x_2)}{x_2} \right)^2. \quad (2.24)$$

Define

$$\begin{aligned} f_1(t_1, t_2, u_1, u_2) &:= t_2^{2M} P_m \left(\frac{u_1^2}{t_2^2}, \frac{16(t_1 - t_2)^2}{t_2^2} \right) \\ f_2(t_1, t_2, u_1, u_2) &:= 2u_2 - t_1. \end{aligned}$$

Similarly to the proof of Lemma 2.6, applying the algebraic independence criterion, we have to show that

$$\frac{\partial P_m}{\partial t_2}(X_0, Y_0) = -\frac{2}{x_2} \left(X_0 \frac{\partial P_m}{\partial X}(X_0, Y_0) + (Y_0 + 4\sqrt{Y_0}) \frac{\partial P_m}{\partial Y}(X_0, Y_0) \right) \neq 0. \quad (2.25)$$

By the hypothesis of Lemma 2.7 the polynomial

$$R(Y) := \text{Res}_X \left(P_m(X^2, Y^2), X^2 \frac{\partial P_m}{\partial X}(X^2, Y^2) + (Y^2 + 4Y) \frac{\partial P_m}{\partial Y}(X^2, Y^2) \right) \in \mathbb{Z}[Y]$$

does not vanish identically. Since the numbers x_1 and x_2 are algebraically independent over \mathbb{Q} , we have $R(Y_1) \neq 0$ for $Y_1 := 4(x_1 - x_2)/x_2$, and hence the polynomials $P_m(X^2, Y_1^2)$ and

$$X^2 \frac{\partial P_m}{\partial X}(X^2, Y_1^2) + (Y_1^2 + 4Y_1) \frac{\partial P_m}{\partial Y}(X^2, Y_1^2)$$

have no common root. Noting that $P_m(X_1^2, Y_1^2) = 0$ holds for $X_1 := y_1/x_2$, we obtain

$$X_1^2 \frac{\partial P_m}{\partial X}(X_1^2, Y_1^2) + (Y_1^2 + 4Y_1) \frac{\partial P_m}{\partial Y}(X_1^2, Y_1^2) \neq 0.$$

Finally, using $X_0 = X_1^2$ and $Y_0 = Y_1^2$ by (2.24), we obtain (2.25), which completes the proof of Lemma 2.7. \square

Lemma 2.8. *If the polynomial*

$$\text{Res}_X \left(P_m(X, Y), X^2 \left(\frac{\partial P_m}{\partial X}(X, Y) \right)^2 - Y(Y - 16) \left(\frac{\partial P_m}{\partial Y}(X, Y) \right)^2 \right)$$

does not vanish identically, then the numbers $\theta_4(n\tau)$ and $\theta_4(\tau)$ are algebraically independent over \mathbb{Q} .

Proof of Lemma 2.8. By (2.13), (2.14), and (2.15), the numbers X_0 and Y_0 in (2.21) are expressed as

$$X_0 = \frac{y_1}{x_2 y_2} \quad \text{and} \quad Y_0 = -4 \frac{(x_2 - y_2)^2}{x_2 y_2}$$

with

$$Y_0(Y_0 - 16) = (\sqrt{Y_0(Y_0 - 16)})^2 := \left(\frac{4(x_2^2 - y_2^2)}{x_2 y_2} \right)^2,$$

where

$$\begin{aligned} x_1 &:= \theta_4^2, & x_2 &:= \theta_3^2, \\ y_1 &:= m^2 \theta_4^4(n\tau), & y_2 &:= \theta_4^2. \end{aligned}$$

Define

$$f_1(t_1, t_2, u_1, u_2) := (t_2 u_2)^M P_m \left(\frac{u_1}{t_2 u_2}, -4 \frac{(t_2 - u_2)^2}{t_2 u_2} \right)$$

$$f_2(t_1, t_2, u_1, u_2) := u_2 - t_1.$$

Then, similarly to the proofs of previous lemmas, we have only to prove

$$\frac{\partial P_m}{\partial t_2}(X_0, Y_0) = -\frac{1}{x_2} \left(X_0 \frac{\partial P_m}{\partial X}(X_0, Y_0) - \sqrt{Y_0(Y_0 - 16)} \frac{\partial P_m}{\partial Y}(X_0, Y_0) \right) \neq 0, \quad (2.26)$$

which follows immediately from the hypothesis of Lemma 2.8. \square

2.2.2 The case $n = 4m$ with odd integer m

Lemma 2.9. *Let $n = 4m$, where $m > 1$ is an odd integer. Let q be an algebraic number with $q = e^{\pi i \tau}$ and $\Im(\tau) > 0$. If the polynomial*

$$\text{Res}_X \left(P_m(X^4, Y^4), X^4 \frac{\partial}{\partial X} P_m(X^4, Y^4) + (Y^4 + 2Y^3) \frac{\partial}{\partial Y} P_m(X^4, Y^4) \right)$$

does not vanish identically, then the numbers $\theta_3(n\tau)$ and $\theta_3(\tau)$ are algebraically independent over \mathbb{Q} .

Proof of Lemma 2.9. Let

$$\begin{aligned} x_1 &:= 2\theta_3, & x_2 &:= \theta_3 + \theta_4, \\ y_1 &:= 16m^2 \theta_3^4(n\tau), & y_2 &:= \theta_3. \end{aligned}$$

Then, by the identities (2.16) and (2.17), the polynomial $P_m(X, Y)$ vanishes at

$$X_0 = \frac{y_1}{x_2^4} \quad \text{and} \quad Y_0 = (\sqrt[4]{Y_0})^4 := \left(\frac{2(x_1 - x_2)}{x_2} \right)^4 \quad \text{with} \quad \sqrt[4]{Y_0^3} := (\sqrt[4]{Y_0})^3.$$

Again we have $2y_2 - x_1 = 0$, and x_1, x_2 are algebraically independent over \mathbb{Q} for any algebraic number $q = e^{\pi i \tau}$ with $\Im(\tau) > 0$. We introduce the polynomials

$$\begin{aligned} f_1(t_1, t_2, u_1, u_2) &:= t_2^{4M} P_m \left(\frac{u_1}{t_2^4}, \frac{16(t_1 - t_2)^4}{t_2^4} \right), \\ f_2(t_1, t_2, u_1, u_2) &:= 2u_2 - t_1. \end{aligned}$$

Using the algebraic independence criterion, we have to show that

$$\frac{\partial f_1}{\partial t_2}(x_1, x_2, y_1, y_2) = -4x_2^{4M-1} \left(X_0 \frac{\partial P_m}{\partial X}(X_0, Y_0) + (Y_0 + 2\sqrt[4]{Y_0^3}) \frac{\partial P_m}{\partial Y}(X_0, Y_0) \right) \neq 0. \quad (2.27)$$

which follows from the hypothesis of Lemma 2.9. \square

3 Proof of Theorems

Proof of Theorem 1.1. We consider the case of $\theta_3(\tau)$. Let

$$\begin{aligned}\alpha_1 &:= 16\theta_2^4, & \alpha_2 &:= \theta_3^4, \\ \beta_1 &:= n^2\theta_3^4(n\tau), & \beta_2 &:= \theta_3^4,\end{aligned}$$

where, by Lemmas 2.1 and 2.3, the numbers α_1 and α_2 are algebraically independent over \mathbb{Q} and the number $\beta_1/\beta_2 = h_3(\tau)$ is transcendental. Define $P(X, Y) := P_n(Y, X)$ and $Q(X, Y) := Y$. By (1.2)

$$P(\alpha_1/\alpha_2, \beta_1/\beta_2) = P_n(h_3(\tau), 16\lambda(\tau)) = 0$$

and

$$Q(\beta_1, \beta_2) = \beta_2 = \alpha_2.$$

Hence, applying Lemma 2.4, we obtain the algebraic independence over \mathbb{Q} of the numbers β_1 and β_2 . This implies that the numbers $\theta_3(n\tau)$ and $\theta_3(\tau)$ are algebraically independent over \mathbb{Q} .

The same holds for the sets $\{\theta_2(n\tau), \theta_2(\tau)\}$ and $\{\theta_4(n\tau), \theta_4(\tau)\}$. In these cases, we use the identities (1.1), (1.3) and Lemma 2.4 with

$$\begin{aligned}\alpha_1 &:= 16(\theta_2^4 - \theta_3^4), & \alpha_2 &:= \theta_2^4, \\ \beta_1 &:= n^2\theta_2^4(n\tau), & \beta_2 &:= \theta_2^4, \\ P(X, Y) &:= P_n(Y, X), & Q(X, Y) &:= Y,\end{aligned}$$

and

$$\begin{aligned}\alpha_1 &:= 16\theta_2^4, & \alpha_2 &:= \theta_2^4 - \theta_3^4, \\ \beta_1 &:= n^2\theta_4^4(n\tau), & \beta_2 &:= \theta_4^4, \\ P(X, Y) &:= P_n(Y, X), & Q(X, Y) &:= -Y,\end{aligned}$$

respectively. In the latter case, we note that the equality $Q(\beta_1, \beta_2) = \alpha_2$ holds from the identity (2.1). Thus, the proof of Theorem 1.1 is completed. \square

Proof of Theorem 1.2. We first consider the case $n = 2$. Let $F := \mathbb{Q}(x_1, x_2)$, where $x_1 := 2\theta_2^2(2\tau)$ and $x_2 := \theta_2^4$. Then by the identity (2.13) together with the relation $\theta_2^4 = \theta_3^4 - \theta_4^4$, we have $F \subset E := \mathbb{Q}(\theta_3, \theta_4)$ and

$$2x_1\theta_3^2 - x_1^2 - x_2 = 2x_1\theta_4^2 + x_1^2 - x_2 = 0.$$

This implies that the field extension E/F is algebraic, so that

$$\text{trans deg } F/\mathbb{Q} = \text{trans deg } E/F + \text{trans deg } F/\mathbb{Q} = \text{trans deg } E/\mathbb{Q} = 2,$$

which implies that the numbers $\theta_2(2\tau)$ and $\theta_2(\tau)$ are algebraically independent over \mathbb{Q} . For $n = 4$, putting $F := \mathbb{Q}(2\theta_2(4\tau), \theta_2^4)$ and using (2.16), we can proceed the same argument as stated above. For $x_1 := 2\theta_2(4\tau)$ and $x_2 := \theta_2^4$ we use the identities

$$\theta_3^4 - (x_1 - \theta_3)^4 - x_2 = \theta_4^4 - (x_1 + \theta_4)^4 + x_2 = 0.$$

In the case of $n = 6$, we use Lemma 2.6. From [8] we know that

$$P_3(X, Y) = 9 - (Y^2 - 16Y + 28)X + 30X^2 - 12X^3 + X^4.$$

Hence, we have

$$\begin{aligned} & \text{Res}_X \left(P_3(X, Y), X \frac{\partial}{\partial X} P_3(X, Y) + 2(Y - 16) \frac{\partial}{\partial Y} P_3(X, Y) \right) \\ &= 9Y(5Y - 512)(Y - 16)^2(Y - 8)^4 \neq 0. \end{aligned}$$

Lemma 2.6 gives the desired result for $n = 6$. \square

Proof of Theorem 1.3. Similarly to the proof of Theorem 1.2, we can deduce the conclusion for the cases $n = 2, 4, 8$ by using the identities (2.15), (2.18), and

$$32\theta_4^8(8\tau) = (\theta_3 + \theta_4)^4 (\theta_3^2 + \theta_4^2) \theta_3 \theta_4,$$

which is yielded from (2.14), (2.15), and (2.18). In the cases of $n = 6, 10$, we use Lemma 2.8. For $n = 6$ we compute the resultant from the lemma explicitly.

$$\begin{aligned} & \text{Res}_X \left(P_3(X, Y), X^2 \left(\frac{\partial P_3}{\partial X}(X, Y) \right)^2 - Y(Y - 16) \left(\frac{\partial P_3}{\partial Y}(X, Y) \right)^2 \right) \\ &= -81Y^3(375Y^2 - 6000Y + 262144)(Y - 16)^3(Y - 8)^8 \neq 0. \end{aligned}$$

Next, let $n = 10$. In [8] the polynomial $P_5(X, Y)$ is given as well.

$$\begin{aligned} P_5(X, Y) &= 25 - (126 - 832Y + 308Y^2 - 32Y^3 + Y^4)X + (255 + 1920Y - 120Y^2)X^2 \\ &\quad + (-260 + 320Y - 20Y^2)X^3 + 135X^4 - 30X^5 + X^6. \end{aligned}$$

Hence, by setting

$$T_{10}(Y) := \text{Res}_X \left(P_5(X, Y), X^2 \left(\frac{\partial P_5}{\partial X}(X, Y) \right)^2 - Y(Y - 16) \left(\frac{\partial P_5}{\partial Y}(X, Y) \right)^2 \right),$$

we obtain $T_{10}(1) \equiv 1 \pmod{2}$, such that $T_{10}(Y)$ does not vanish identically. This completes the proof of Theorem 1.3. \square

Proof of Theorem 1.4. Taking the results from Theorem A and Theorem 1.1 into account, for Theorem 1.4 it suffices to consider $n \in \{14, 18, 20, 22\}$. Here, we compute the resultants from Lemma 2.7 (for $n \in \{14, 18, 22\}$) and from Lemma 2.9 (for $n = 20$) explicitly by using a computer algebra system. In order to show that the resultants do not vanish we again consider the values at $Y = 1$. For $n = 14$ we use

$$\begin{aligned} P_7(X, Y) &= 49 - (344 - 17568Y + 20554Y^2 - 6528Y^3 + 844Y^4 - 48Y^5 + Y^6)X \\ &\quad + (1036 + 156800Y + 88760Y^2 - 12320Y^3 + 385Y^4)X^2 \\ &\quad - (1736 - 185024Y + 18732Y^2 - 896Y^3 + 28Y^4)X^3 \\ &\quad + (1750 + 31360Y - 1960Y^2)X^4 - (1064 - 2464Y + 154Y^2)X^5 \\ &\quad + 364X^6 - 56X^7 + X^8 \end{aligned}$$

(cf. [3]) to obtain the resultant from Lemma 2.7,

$$T_{14}(Y) := \text{Res}_X \left(P_7(X^2, Y^2), X^2 \frac{\partial}{\partial X} P_7(X^2, Y^2) + (Y^2 + 4Y) \frac{\partial}{\partial Y} P_7(X^2, Y^2) \right).$$

It follows that $T_{14}(1) \equiv 1 \pmod{2}$. This shows that $T_{14}(1) \neq 0$. For $n = 18$ we apply Lemma 2.7 with

$$\begin{aligned} P_9(X, Y) = & 6561 - (60588 - 18652032Y + 56033208Y^2 - 40036032Y^3 + 11743542Y^4 \\ & - 1715904Y^5 + 132516Y^6 - 5184Y^7 + 81Y^8)X \\ & + (250146 + 427613184Y + 2083563072Y^2 + 86274432Y^3 - 57982860Y^4 \\ & + 4249728Y^5 - 99288Y^6 + 576Y^7 - 9Y^8)X^2 \\ & - (607420 - 1418904064Y + 2511615520Y^2 - 353755456Y^3 + 19071754Y^4 \\ & - 612736Y^5 + 13960Y^6 - 64Y^7 + Y^8)X^3 \\ & + (959535 + 856286208Y + 8468928Y^2 - 2145024Y^3 - 808488Y^4 \\ & + 65664Y^5 - 1368Y^6)X^4 \\ & - (1028952 + 22899456Y + 1430352Y^2 - 505152Y^3 + 38826Y^4 \\ & - 1728Y^5 + 36Y^6)X^5 \\ & + (757596 - 13138944Y + 4160448Y^2 - 417408Y^3 + 13044Y^4)X^6 \\ & - (378072 + 1138176Y + 16416Y^2 - 10944Y^3 + 342Y^4)X^7 \\ & + (122895 + 64512Y - 4032Y^2)X^8 - (24060 - 11136Y + 696Y^2)X^9 \\ & + 2466X^{10} - 108X^{11} + X^{12}. \end{aligned}$$

We have

$$T_{18}(Y) := \text{Res}_X \left(P_9(X^2, Y^2), X^2 \frac{\partial}{\partial X} P_9(X^2, Y^2) + (Y^2 + 4Y) \frac{\partial}{\partial Y} P_9(X^2, Y^2) \right),$$

and thus $T_{18}(1) \equiv 1 \pmod{2}$. Hence, $T_{18}(1) \neq 0$. For $n = 20$ we need the polynomial $P_5(X, Y)$, which was already used in the proof of Theorem 1.3. The resultant from Lemma 2.9,

$$T_{20}(Y) := \text{Res}_X \left(P_5(X^4, Y^4), X^4 \frac{\partial}{\partial X} P_5(X^4, Y^4) + (Y^4 + 2Y^3) \frac{\partial}{\partial Y} P_5(X^4, Y^4) \right),$$

satisfies $T_{20}(1) \equiv 1 \pmod{2}$. Finally, for $n = 22$ we again apply Lemma 2.7 with

$$\begin{aligned} P_{11}(X, Y) = & 121 - (1332 - 2214576Y + 15234219Y^2 - 21424896Y^3 + 11848792Y^4 \\ & - 3309152Y^5 + 522914Y^6 - 48896Y^7 + 2684Y^8 - 80Y^9 + Y^{10})X \\ & + (6666 + 111458688Y + 2532888424Y^2 + 2367855776Y^3 - 327773413Y^4 \\ & - 9982720Y^5 + 3230480Y^6 - 161920Y^7 + 2530Y^8)X^2 \\ & - (20020 - 864654912Y + 12880909668Y^2 - 5289254784Y^3 + 744094076Y^4 \\ & - 43914992Y^5 + 967461Y^6 - 2816Y^7 + 44Y^8)X^3 \end{aligned}$$

$$\begin{aligned}
& + (40095 + 1748954240Y - 175142088Y^2 + 372281536Y^3 - 68516998Y^4 \\
& + 4266240Y^5 - 88880Y^6)X^4 \\
& - (56232 - 1061669664Y + 132688050Y^2 - 10724736Y^3 + 715308Y^4 \\
& - 28512Y^5 + 594Y^6)X^5 \\
& + (56364 + 211953280Y - 7454568Y^2 - 724064Y^3 + 22627Y^4)X^6 \\
& - (40392 - 24140864Y + 2162116Y^2 - 81664Y^3 + 2552Y^4)X^7 \\
& + (20295 + 1448832Y - 90552Y^2)X^8 - (6820 - 36784Y + 2299Y^2)X^9 \\
& + 1386X^{10} - 132X^{11} + X^{12}.
\end{aligned}$$

Here, we obtain

$$T_{22}(Y) := \text{Res}_X \left(P_{11}(X^2, Y^2), X^2 \frac{\partial}{\partial X} P_{11}(X^2, Y^2) + (Y^2 + 4Y) \frac{\partial}{\partial Y} P_{11}(X^2, Y^2) \right),$$

where $T_{22}(1) \equiv 3 \pmod{13}$, such that $T_{22}(1) \neq 0$. The proof of Theorem 1.2 is complete. \square

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